Computing Lyapunov Exponents of Hybrid Dynamical Systems

Joseph Dinius

***Abstract*— This paper attempts to merge the theory of Lyapunov exponents for continuous- and discrete-time dynamical systems for application to mixed, or hybrid, dynamical systems. The relevant theoretical background is presented and extended to hybrid dynamical systems. A method for computing Lyapunov exponents is presented and applied to the problem of colliding particles.**

I. INTRODUCTION

In complex dynamical systems, microscopic properties such as position are oftentimes distributed in such a way that classical methods of analysis are not practical. This is, in general, due to the effect of small perturbations on long-term system behavior. Instead of analyzing localized properties, it is most useful to compute global properties that are indicative of a system's macroscopic behavior. But what types of global properties are of interest? For most dynamical systems, it is natural to wonder how two trajectories that start very close will evolve in time. Will they stay close, or will they diverge? How quickly do these trajectories converge or diverge? Answering these questions requires the computation of the so-called *Lyapunov exponents*, which describe the average rate of exponential divergence (or convergence) of two neighboring trajectories.

A very large body of work has been done concerning the computation of Lyapunov exponents for continuous- and discrete-time dynamics [3]. However, little work has been done combining analytical and computational methods of both approaches to address hybrid dynamical systems. In practice, there are some difficulties in merging methods used for the two approaches; however, using simple geometric considerations, analytical and computational methods can be derived for computing the Lyapunov exponents of hybrid dynamical systems. The focus of this work is the derivation of such methods.

This paper begins by developing the necessary theoretical background for computing Lyapunov exponents. Subsequently, the theory of Lyapunov exponents will be used to develop a coherent framework for application to hybrid dynamical systems. After developing this framework, the theoretical approach is applied to the problem of colliding particles; a system that evolves continuously in time until collisions instantaneously change the velocity of the colliding particles.

II. THEORY

This section is broken up in subsections. Continuous-time dynamics are considered first, from which the same concepts will be applied to discrete-time dynamics. The main points of both subsections will be combined for application to hybrid dynamical systems. Subsequently, the numerical procedure for computing Lyapunov exponents of hybrid dynamical systems will be developed.

A. Continuous Dynamical Systems

Consider the N-dimensional continuous dynamical system with flow set $C \subseteq \mathbb{R}^N$ described by the system of first-order differential equations

$$\dot{\mathbf{x}} = f(\mathbf{x}). \tag{1}$$

If the system is perturbed slightly at initialization,

This work was completed for A ME549 at the University of Arizona, Fall 2011

J. Dinius is a graduate student in the Program in Applied Mathematics, University of Arizona, Tucson, AZ 85721, USA `jdinius@math.arizona.edu`

the dynamics become

$$\begin{aligned}
\dot{\mathbf{x}} + \dot{\delta\mathbf{x}} &= (\mathbf{x} \dot{+} \delta\mathbf{x}) \\
&= f(\mathbf{x} + \delta\mathbf{x}) \\
&\approx f(\mathbf{x}) + \mathcal{D}f(\mathbf{x})\delta\mathbf{x}
\end{aligned}$$

where

$$\mathcal{D}f(\mathbf{x}) = \frac{\partial f}{\partial \mathbf{x}}(\mathbf{x})$$

is the Jacobian matrix of f evaluated at the point in phase space $\mathbf{x}$. The evolution equation for the perturbation can now be derived as

$$\begin{aligned}
\dot{\delta\mathbf{x}} &= \dot{\mathbf{x}} + \dot{\delta\mathbf{x}} - \dot{\mathbf{x}} \\
&= (\mathbf{x} \dot{+} \delta\mathbf{x}) - \dot{\mathbf{x}} \\
&\approx f(\mathbf{x}) + \mathcal{D}f(\mathbf{x})\delta\mathbf{x} - f(\mathbf{x}) \\
&= \mathcal{D}f(\mathbf{x})\delta\mathbf{x}.
\end{aligned}$$

Through the use of separation-of-variables, the differential equation in $\delta\mathbf{x}$ will have a solution that looks like

$$\delta\mathbf{x}_m(t) = \delta\mathbf{x}_m(0)e^{\overline{\lambda}_m t} \tag{2}$$

with

$$\delta\mathbf{x}_m(0) = \sum_{i=1}^{N} a_{im}\mathbf{e}_i. \tag{3}$$

In (2) and (3), the following is assumed

1) $\{\mathbf{e}_i | 1 \leq i \leq N\}$ is the set of standard basis vectors on $\mathbb{R}^N$.
2) The set $\{\delta\mathbf{x}_m(0) | \ 1 \leq m \leq N\}$ is an orthonormal set that spans the tangent space of C (the space of all perturbations to a solution of (1), denoted δC). Consequently, $||\mathbf{a}_m||_2 = 1$ where $\mathbf{a}_m(\in \mathbb{R}^N) = [a_{1m} \ldots a_{Nm}]$.
3) $|| \cdot ||_2$ denotes the standard Euclidean norm on $\mathbb{R}^N$.

Here, $\delta\mathbf{x}_m(t)$ denotes the time-evolution of $\delta\mathbf{x}_m(0)$. The $\overline{\lambda}_m$ above will be the m^{th} eigenvalue of $\int_0^t \mathcal{D}f(\mathbf{x})d\tau$ $(1 \leq m \leq N)$ which, along with the corresponding eigenvector, is highly localized in both time and space. Instead of focusing on this localized relationship, it is more useful to consider the *time-averaged* quantity of $\overline{\lambda}_m$:

$$\lambda_m = \lim_{t\to\infty} \overline{\lambda}_m = \lim_{t\to\infty} \frac{1}{t} \log\left(\frac{||\delta\mathbf{x}_m(t)||_2}{||\delta\mathbf{x}_m(0)||_2}\right), \tag{4}$$

λ_m is called the m^{th} *Lyapunov exponent* of the dynamical system. At this point, a number of mathematically relevant questions should be addressed. Does the limit in (4) converge? Assuming it does converge for each initial condition of the unperturbed system, does it converge to the same value for each initial condition? The system has N dimensions, which direction is the growth attributed to λ_m in? Is it guaranteed that any initial choice of $\delta\mathbf{x}_m$ will yield the same calculation of λ_m?

First, address the limit question. Oseledec's multiplicative ergodic theorem [1] guarantees that, under very general assumptions, the limit (4) exists. Ruelle [6] has shown that, again under very general assumptions, (4) converges to the same value regardless of initial starting point. This indicates the *global* nature of Lyapunov exponents in analyzing stability properties of a dynamical system.

Answering the question of direction is much trickier. In general, this direction is time-dependent. The reason for this is that the directions of growth are localized in time, like the eigenvectors/eigenvalues of the Jacobian at a given point along the trajectory $\mathbf{x}$. However, the Lyapunov exponents represent a global average. This leads to another potential issue, and that is: How does one choose an initial perturbation to compute the m^{th} Lyapunov exponent? From Ott [3], the choice of any initial perturbation will allow the computation of the largest Lyapunov exponent, denoted λ_1. To compute subsequent $\lambda_2, \ldots, \lambda_N$, let $E_m \subset \delta C$ denote the subspace orthogonal to the space spanned by the initial perturbation vectors $\delta\mathbf{x}_1(0), \ldots, \delta\mathbf{x}_{m-1}(0)$. Pick $\delta\mathbf{x}_m(0) \in E_m$ and then compute λ_m via (4). Back to the question: Is it guaranteed that any initial choice of $\delta\mathbf{x}_m$ will yield the same calculation of λ_m? From the preceding construction, so long as $\delta\mathbf{x}_m(0)$ is chosen so that it is orthogonal to all $\delta\mathbf{x}_1(0), \ldots, \delta\mathbf{x}_{m-1}(0)$, the same value for λ_m will be computed. Using this approach in practice, however, is intractable, as large dynamical systems can rarely be analyzed analytically with ease. The approach outlined can be implemented numerically to compute all Lyapunov exponents using the QR factorization method known as Gram-Schmidt reorthonormalization [3], [5]. This method will be developed in the Numerical Procedure section.

B. Discrete Dynamical Systems

Much of the construction of Lyapunov exponents for discrete-time dynamical systems follows directly, so the development will be terse. Consider the N-dimensional discrete dynamical system with domain $D \subseteq \mathbb{R}^n$ described by the map

$$\mathbf{x}^+ = g(\mathbf{x}). \tag{5}$$

If the system is perturbed slightly at initialization,

$$\begin{aligned} \mathbf{x}^+ + \delta\mathbf{x}^+ &= (\mathbf{x} + \delta\mathbf{x})^+ \\ &= g(\mathbf{x} + \delta\mathbf{x}) \\ &\approx g(\mathbf{x}) + \mathcal{D}g(\mathbf{x})\delta\mathbf{x} \end{aligned}$$

where

$$\mathcal{D}g(\mathbf{x}) = \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x})$$

is the Jacobian matrix of g evaluated at the point $\mathbf{x}$. The evolution equation for the perturbation is

$$\begin{aligned} \delta\mathbf{x}^+ &= \mathbf{x}^+ + \delta\mathbf{x}^+ - \mathbf{x}^+ \\ &= (\mathbf{x} + \delta\mathbf{x})^+ - \mathbf{x}^+ \\ &\approx g(\mathbf{x}) + \mathcal{D}g(\mathbf{x})\delta\mathbf{x} - g(\mathbf{x}) \\ &= \mathcal{D}g(\mathbf{x})\delta\mathbf{x}. \end{aligned}$$

Following almost exactly the same approach as used in the continuous case, with the same assumptions, it is possible to derive the average rates (one for each dimension of the system) of growth of $\delta\mathbf{x}$,

$$\lambda_m = \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\|(\delta\mathbf{x}_m^+)^n\|_2}{\|\delta\mathbf{x}_m\|_2}\right). \tag{6}$$

Here, $(\delta\mathbf{x}_m^+)^n$ denotes the application of the Jacobian of the map g to the perturbation $\delta\mathbf{x}_m$ n times. The rest of the arguments concerning the calculation of the Lyapunov exponents for discrete-time systems follow exactly from those made in the Continuous-Time section.

C. Hybrid Dynamical Systems

Consider the N-dimensional hybrid dynamical system $\mathcal{H}$ with state variable $\mathbf{x} \in \mathbb{R}^N$ described by the sets $C \subseteq \mathbb{R}^N$ and $D \subseteq \mathbb{R}^N$ and the real-valued functions f and g. The sets C and D denote the flow and jump sets, respectively. The flow set is defined as the set where the dynamics of $\mathcal{H}$ are continuous in time, while the jump set is defined as the set where the dynamics of $\mathcal{H}$ are discrete. The function f describes the time-derivative of the state variable $\mathbf{x}$ for $\mathbf{x} \in C$. The function g describes the mapping of points in D. This mapping can be such that $g(\mathbf{x})$ remains in D, or jumps out of D and into the flow set C. For a more thorough treatment regarding the construction of $\mathcal{H}$, see [7].

It is desirable to try and combine the constructions for Lyapunov exponents from the preceding sections, however some care must be taken. When perturbing a trajectory, the times when $\mathcal{H}$ flows or jumps will be offset by some factor. Figure 1 shows the two-dimensional projection of two trajectories representing solutions to $\mathcal{H}$: unperturbed (solid) and perturbed (dashed). Due to the initial offset, the times at which each trajectory will enter the jump set are different. Each perturbed solution will have different hybrid time domain than the original reference trajectory. Keeping track of all of these different time domains would be difficult; however, it is possible to make a modification to the jump map incorporating the time offset $\delta\tau_m$ of the j^{th} time the trajectory $\mathbf{x}$ jumps so that only the hybrid time domain of the reference trajectory needs to be considered. Therefore, the continuous-time construction for the evolution of the perturbation vectors is valid and can be used as is. Only the evolution of the discrete-time perturbations needs to be modified.

Consider the evolution of the two trajectories shown in Figure 1. It is desired to derive a map S that will describe the mapping of a perturbation vector after a jump occurs in the reference trajectory. From the figure, it is observed that (recalling (5))

$$\delta\mathbf{x}^+ = g(\mathbf{x} + \delta\mathbf{x}_m) - [\mathbf{x}^+ + f(\mathbf{x}^+)\delta\tau_m] \tag{7}$$

since $\delta\mathbf{x}_m = \delta\mathbf{x} + f(\mathbf{x})\delta\tau_m$. Using the linear approximation utilized in the preceding section on discrete systems, it follows that

$$g(\mathbf{x} + \delta\mathbf{x}_m) = g(\mathbf{x}) + \mathcal{D}g(\mathbf{x})\delta\mathbf{x}_m. \tag{8}$$

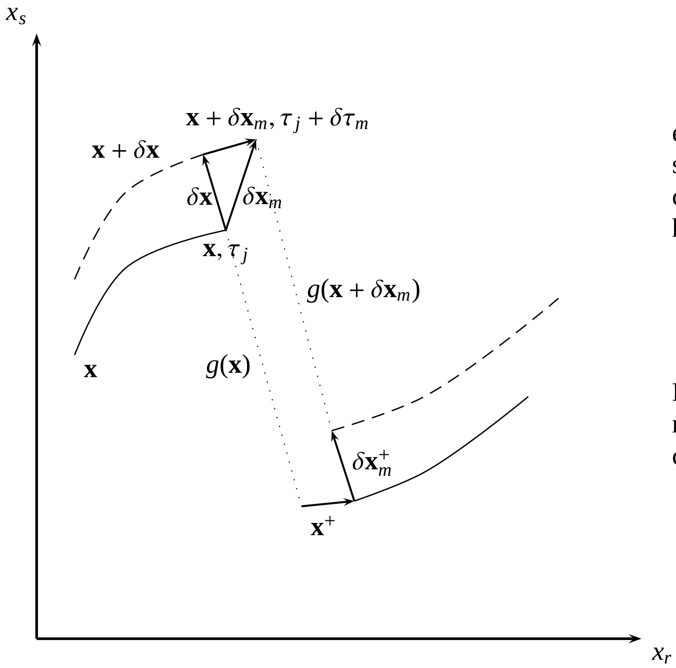

Fig. 1. Effect of time offset on perturbation vectors. $\delta\tau_m$ represents the time difference between the application of the discontinuous map g in the reference (unperturbed) and perturbed trajectories at the time of the j^{th} jump in the reference trajectory (denoted τ_m). In this figure, the choice of $1 \le r, s \le N$, N the dimension of the dynamical system, is arbitrary. The figure represents the application of the flow up to and after the j^{th} jump occurs in the system.

Combining (7) and (8) and utilizing the fact that $\mathbf{x}^+ = g(\mathbf{x})$ yields the final expression for S (see [2], [4] for a thorough treatment):

$$\delta\mathbf{x}_m^+ = S(\mathbf{x}, \delta\mathbf{x}_m) \tag{9}$$

$$:= \mathcal{D}g(\mathbf{x})\delta\mathbf{x}_m + [\mathcal{D}g(\mathbf{x})f(\mathbf{x}) - f \circ g(\mathbf{x})]\delta\tau_m. \tag{10}$$

The (real-valued) time offset can be either positive or negative in this construction, and is generally a function of both $\mathbf{x}$ and $\delta\mathbf{x}$ [2], [4]. Now that the relevant mapping has been derived for perturbations in the setting of hybrid dynamical systems, the construction of Lyapunov exponents can be formalized.

The construction will proceed using notation, slightly modified where necessary, from [7]. Let ϕ be a hybrid arc that is a solution to $\mathcal{H}$. In the preceding sections, Lyapunov exponents have been constructed as averages as t or n become arbitrarily large. Therefore, for the idea of Lyapunov exponents to make sense in the setting of hybrid systems, it is required that ϕ be both maximal and complete. To further this notion, let E denote the hybrid time domain of ϕ,

$$E = \bigcup_{j=0}^{J-1}\Big([\tau_j, \tau_{j+1}], j\Big).$$

For ϕ to be complete (and hence, maximal), it is necessary and sufficient that one of the following conditions holds:

1) If at least one flow interval exists and the system flows from the finite time τ_{J-1} up to an arbitrarily large time, then $J < \infty$ and the last flow interval will be $[\tau_{J-1}, \infty)$.
2) If at least one flow interval exists and the system flows from the finite time τ_{j-1} up to the finite time τ_j, and for sufficiently large j, $\tau_{j-1} = \tau_j$, then $J = \infty$.
3) If at least one flow interval exists, and, for all j, the interval $[\tau_j, \tau_{j+1}]$ is dense in $\mathbb{R}_{\ge 0}$, then $J = \infty$.
4) If $J = 0$, then the system flows from $t_0 = 0$ to an arbitrarily large time T; that is the interval $[0, \infty)$.
5) If $\tau_j = \tau_k$ for all $j, k \le J$, then $J = \infty$.

One of the first three properties is required if $\mathcal{H}$ has both continuous- and discrete-time dynamics. Property 4 is required if $\mathcal{H}$ is continuous, while property 5 is required if $\mathcal{H}$ is discrete. Property 1 is interpreted as the case where the system alternates between flows and jumps until a final condition is reached where the system state can no longer enter the jump set. Property 2 is interpreted as the case where the system flows until the system state reaches a condition where it can only subsequently jump. Property 3 is the case where the system alternates between flows and jumps for all time.

To construct the Lyapunov exponents for a hybrid system, the notion of solution needs to be extended to include the evolution of the perturbation. Since ϕ is a solution to $\mathcal{H}$, it follows that for all $j \in \mathbb{N}$ such that $I^j := \{t : (t, j) \in \operatorname{dom}\phi\}$ has nonempty interior,

$$\begin{cases} \phi(t, j) \in C & \text{for all } t \in \operatorname{int} I^j, \\ \dot{\phi}(t, j) \in F(\phi(t, j)) & \text{for almost all } t \in I^j; \end{cases} \tag{11}$$

and for all $(t, j) \in \mathrm{dom}\,\phi$ such that $(t, j+1) \in \mathrm{dom}\,\phi$,

$$\begin{cases} \phi(t, j) \in D, \\ \phi(t, j+1) \in G(\phi(t, j)). \end{cases} \tag{12}$$

To include the continuous-time perturbation dynamics, augment (11) with the condition that (given that $\delta\phi(t, j)$ is the evolution of the perturbation)

$$\dot{\delta\phi}(t, j) \in \mathcal{D}F(\phi(t, j), \delta\phi(t, j)) \text{ for a.e. } t \in \text{int } I^j;$$

where $\mathcal{D}F(\phi(t, j), \delta\phi(t, j)) := \{\mathcal{D}f(\phi(t, j), \delta\phi(t, j))\delta\phi(t, j) |\ t \in \text{int } I^j\}$. Similarly, for the discrete-time dynamics, augment (12) with the condition that

$$\delta\phi(t, j+1) \in \mathcal{S}(\phi(t, j), \delta\phi(t, j));$$

where $\mathcal{S} := \{S(\phi(t, j), \delta\phi(t, j)) |$ for all $(t, j) \in \mathrm{dom}\,\phi$ such that $(t, j+1) \in \mathrm{dom}\,\phi\}$. Now, it makes sense to construct a relationship for the Lyapunov exponents similar to those constructed in the preceding sections:

$$\lambda_m = \lim_{s \to \infty} \frac{1}{s} \log\left(\frac{||\delta\phi(t, j)||_2}{||\delta\phi_m(0, 0)||_2}\right).$$

The quantity s will be the larger of the two suprema of the hybrid time domain E; that is,

$$s := \max(\sup_t E, \sup_j E).$$

Notice that, in the event that $\mathcal{H}$ contains strictly continuous or discrete dynamics, expressions (4) and (6) are recovered. Concerning the initial perturbation $\delta\phi_m(0, 0)$: This initial offset can be chosen arbitrarily so long as $\phi(0, 0) + \delta\phi_m(0, 0) \in \overline{C} \cup D$ for all $1 \le m \le N$. To compute the N Lyapunov exponents, follow the same process outlined in the preceding sections.

D. Numerical Procedure

As stated previously, the analytical computation of the Lyapunov exponents for an arbitrary dynamical system would be a rather difficult task. The rough outline of an algorithm provided previously is an analog of the QR factorization process known as Gram-Schmidt reorthonormalization. Due to numerical instability in the original Gram-Schmidt process, the modified Gram-Schmidt process is typically used. Essentially, the procedure implemented should start with a set of perturbation vectors that span $\mathbb{R}^n$. The dynamics of both the reference trajectory ϕ and all of the N $\delta\phi$'s should be integrated/iterated at each time step. After each integration/iteration step, apply modified Gram-Schmidt to obtain the normalization factors (the logarithm of which are the Lyapunov exponents) and the directions of growth. Algorithm 1 shows pseudocode for this procedure:

Algorithm 1 Computing Lyapunov Exponents with Modified Gram-Schmidt

> Initialize $\phi(0, 0)$ and $\delta\phi_m(0, 0)$ for all $1 \le m \le N$. Choose $\delta\phi_m(0, 0) = \mathbf{e}_m$, the standard basis vector in $\mathbb{R}^n$ along the m^{th} coordinate.
> **while** $(\max(t, j) < M)$ where M is a prescribed stopping point of the integration/iteration of solution to $\mathcal{H}$ **do**
>   Integrate/iterate $\phi(t, j)$ and all of the $\delta\phi_m(t, j)$
>   Perform Modified Gram-Schmidt process
>   Increment time t and/or iteration counter j
> **end while**

Algorithm 2 describes the Modified Gram-Schmidt algorithm in detail:

Algorithm 2 Modified Gram-Schmidt Reorthonormalization MGSR($\{\delta\boldsymbol{\Gamma}_i\}$)

> $\{\delta\phi(t, j)'_i\} \leftarrow \{\delta\phi(t, j)_i\}$
> **for** $i \leftarrow 1 : N$ **do**
> $\delta\phi(t, j)'_i \leftarrow \frac{\delta\phi(t, j)_i}{||\delta\phi(t, j)_i||}$
>   **for** $k \leftarrow i + 1 : N$ **do**
> $\delta\phi(t, j)_k \leftarrow \delta\phi(t, j)_k - (\delta\phi(t, j)'_i \cdot \delta\phi(t, j)_k)\delta\phi(t, j)'_i$
>   **end for**
> **end for**
> **return** $\{\delta\phi(t, j)'_i\}$

Looking at Algorithm 2 in detail, it is clear how the process is related to the outline presented in the Continuous Dynamical Systems section regarding the computation of all of the Lyapunov exponents. Initially, there is one vector considered (recall that any initial vector will do), then a vector is constructed that is orthogonal to the original vector. On the next iteration, construct a vector that is orthogonal to all previous vectors. Repeat the process a total of N times for an N-dimensional system to obtain all of the Lyapunov exponents. For more detail, see [5].

III. Application: Colliding Particles

A. Mathematical Details

As an application of the method prescribed in the Numerical Procedures section, consider the system of N_{disks} particles that undergo elastic collisions in two spatial dimensions with no external force and subjected to periodic boundary conditions. Periodic boundary conditions mean that the particles are confined to a box of size $[L_x, L_y]$, where L_x is the length of the box in the x-direction, and L_y the length in the y-direction. Once a particle exits the box, its image enters the box from the opposite side of where it exited with the same velocity. There is a large body of work considering this system [2], [4], therefore only a cursory overview is provided. Recall that in an elastic collision, both kinetic energy and linear momentum are conserved. The $4N_{\text{disks}}$-dimensional vector describing the state is

$$\mathbf{x} = [\mathbf{q}_1 \ \mathbf{q}_2 \ \ldots \ \mathbf{q}_{N_{\text{disks}}} \ \mathbf{p}_1 \ \ldots \ \mathbf{p}_{N_{\text{disks}}}]^T,$$

where $\mathbf{q}_j = [x^j \ y^j]^T$ denotes the position of the center of particle j in the xy plane and $\mathbf{p}_j = [p_x^j \ p_y^j]^T$ the linear momentum of particle j. The hybrid system that describes the dynamics of these colliding particles is:

$$\mathcal{H} := \begin{cases} C = (([0 \ L_x] \times [0 \ L_y])^{2N_{\text{disks}}} \times \mathbb{R}^{2N_{\text{disks}}}) \setminus D \\ f(\mathbf{x}) = \begin{pmatrix} \frac{\mathbf{P}}{m} \\ \mathbf{0} \end{pmatrix} \\ D = \Big\{ \mathbf{x} \in (([0 \ L_x] \times [0 \ L_y])^{2N_{\text{disks}}} \times \mathbb{R}^{2N_{\text{disks}}}) \Big| \\ \quad \min_{1 \le i < j \le N_{\text{disks}}} (\|\mathbf{q}_i - \mathbf{q}_j\|_2) \le \sigma \Big\} \\ g(\mathbf{x}) = \mathbf{x}^+ \end{cases}$$

where m is the mass of each particle (assumed equal), $\mathbf{P} = [\mathbf{p}_1 \ \ldots \ \mathbf{p}_{N_{\text{disks}}}]^T$, σ is the diameter of each particle (assumed equal) and $\mathbf{x}^+ = [\mathbf{q}_1^+ \ \mathbf{q}_2^+ \ \ldots \ \mathbf{q}_{N_{\text{disks}}}^+ \ \mathbf{p}_1^+ \ \ldots \ \mathbf{p}_{N_{\text{disks}}}^+]^T$. The particles cannot continue to flow once they have come in contact with another particle, therefore the set C must be open. Assuming that the colliding particles have indices j and k, then

$$\begin{aligned} \mathbf{q}_j^+ &= \mathbf{q}_j \\ \mathbf{p}_j^+ &= \mathbf{p}_j \text{ for all } j \neq k, l \\ \mathbf{p}_k^+ &= \mathbf{p}_k + (\mathbf{p} \cdot \mathbf{q})\mathbf{q}/\sigma^2 \\ \mathbf{p}_l^+ &= \mathbf{p}_l - (\mathbf{p} \cdot \mathbf{q})\mathbf{q}/\sigma^2. \end{aligned}$$

$\mathbf{q}$ and $\mathbf{p}$ are the relative position and momentum of the colliding particles; $\mathbf{q} = \mathbf{q}_l - \mathbf{q}_k$ and $\mathbf{p} = \mathbf{p}_l - \mathbf{p}_k$. The above relationships are a consequence of conservation of linear momentum and kinetic energy [2], [4].

To compute the Lyapunov exponents, construct the evolution equations for the set of perturbation vectors
$\{\delta\mathbf{x}_m | \ 1 \le m \le 4N_{\text{disks}}\}$. During periods of flow in the reference trajectory, a perturbation vector evolves as

$$\dot{\delta\mathbf{x}} = \begin{pmatrix} \frac{\delta\mathbf{P}}{m} \\ \mathbf{0} \end{pmatrix},$$

where $\delta\mathbf{P} = [\delta\mathbf{p}_1 \ \ldots \ \delta\mathbf{p}_{N_{\text{disks}}}]^T$. From (10), it follows that the jump map for perturbation vectors is [2], [4]

$$S(\mathbf{x}, \delta\mathbf{x}) = \delta\mathbf{x}^+$$

where $\delta\mathbf{x}^+ = [\delta\mathbf{q}_1^+ \ \delta\mathbf{q}_2^+ \ \ldots \ \delta\mathbf{q}_{N_{\text{disks}}}^+ \ \delta\mathbf{p}_1^+ \ \ldots \ \delta\mathbf{p}_{N_{\text{disks}}}^+]^T$ and

$$\begin{aligned} \delta\mathbf{q}_j^+ &= \delta\mathbf{q}_j \text{ for all } j \neq k, l \\ \delta\mathbf{p}_j^+ &= \delta\mathbf{p}_j \text{ for all } j \neq k, l \\ \delta\mathbf{q}_k^+ &= \delta\mathbf{q}_k + (\mathbf{q} \cdot \delta\mathbf{q})\delta\mathbf{q}/\sigma^2 \\ \delta\mathbf{q}_l^+ &= \delta\mathbf{q}_j - (\mathbf{q} \cdot \delta\mathbf{q})\delta\mathbf{q}/\sigma^2 \\ \delta\mathbf{p}_k^+ &= \mathbf{p}_k + \frac{1}{\sigma^2}[(\delta\mathbf{p} \cdot \mathbf{q} + \delta\mathbf{q}_c \cdot \mathbf{p})\mathbf{q} + (\mathbf{q} \cdot \mathbf{p})\delta\mathbf{q}_c] \\ \delta\mathbf{p}_l^+ &= \mathbf{p}_l - \frac{1}{\sigma^2}[(\delta\mathbf{p} \cdot \mathbf{q} + \delta\mathbf{q}_c \cdot \mathbf{p})\mathbf{q} + (\mathbf{q} \cdot \mathbf{p})\delta\mathbf{q}_c]; \end{aligned}$$

where $\delta\mathbf{q}_c$ represents the change in the relative position of the two colliding particles due to the time difference between collisions in the reference and perturbed trajectories and is equal to

$$\begin{aligned} \delta\mathbf{q}_c &= \delta\mathbf{q} + \frac{\mathbf{p}}{m}\delta\tau_c \\ \delta\tau_c &= -\frac{\delta\mathbf{q} \cdot \mathbf{q}}{\mathbf{p}/m \cdot \mathbf{q}}. \end{aligned}$$

In all of the above constructions, $\delta\mathbf{q} = \delta\mathbf{q}_l - \delta\mathbf{q}_k$ and $\delta\mathbf{p} = \delta\mathbf{p}_l - \delta\mathbf{p}_k$ are the relative position and momentum coordinates of the perturbation of the colliding particles. Geometrically, $\delta\tau_c$ is interpreted as the offset of the relative position of the colliding particles due to perturbation divided by the relative velocity in the direction of the relative position vector at the time of collision in the reference trajectory.

A few things are worth mentioning before moving on to the Simulation section: It is assumed that, since energy is conserved, the system will alternate between flowing and jumping for all time ($t \to \infty$ and $J \to \infty$). Therefore, the construction of the Lyapunov exponents is meaningful whether choosing to take the limit with respect to time or the number of jumps. For simplicity, the limit will be taken with respect to time.

B. Simulation

Table I shows the simulation parameters used to generate the data presented in this section:

Variable Description	Identifier	Value
Number of Disks	N_{disks}	4
Stopping Time	t_{stop}	1000.0
Density	ρ	0.7
Aspect Ratio	A	$\frac{\sqrt{3}}{2}$
Disk Diameter	σ	1
Kinetic Energy	K	4
Mass	m	1
Timestep	dt	0.001
Normalization	n_{gs}	10

TABLE I

Parameters for simulating colliding particles. Choice of ρ, σ and A determine the dimensions of the simulation box. The aspect ratio A is the length of the box along the x-axis divided by the length of the box along the y-axis. The time step dt is used for output and normalization. The normalization parameter represents the number of time steps of system evolution performed between successive applications of modified Gram-Schmidt.

For further details regarding simulation numerics and setup, see [2], [4], [5]. The expected outcome of these simulations (from physical and mathematical considerations) are as follows:

1) Due to conserved quantities, there should be $2d+2$ Lyapunov exponents that are 0, where d is the number of spatial dimensions (in this case, $d = 2$). The conserved quantities ensure that there are directions in the tangent space (the space of all perturbations) that do not grow or shrink exponentially. The conserved quantities are conservation of kinetic energy (contributes 1 vanishing exponent), total momentum (contributes 2), and center of mass spatial location (contributes 2). Additionally, there must be a zero exponent due to a perturbation in the direction of the reference trajectory flow.
2) Due to Oseledec [1], Ruelle [6] and Benettin [5], the full spectrum of Lyapunov exponents should converge over time.
3) Due to the conservation of energy, the sum of the Lyapunov exponents should be zero [2], [4].

Figure 2 shows the numerical results of the simulation performed.

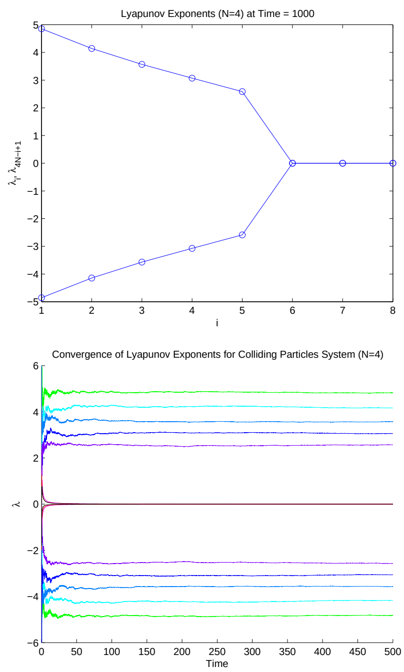

Fig. 2. At top: the Lyapunov spectrum at $t_{\text{stop}} = 1000.0$. At bottom: the Lyapunov spectrum as a function of time. The figure at top demonstrates confirmation of properties 1 and 3 above. The figure at bottom demonstrates confirmation of property 2.

These results compare favorably with those presented in [2], [4]. The exponents converge in a similar time-scale to values that match to within a few percent. As mentioned previously, the convergence of Lyapunov exponents is shows as a function of time since the problem of colliding particles is in category 3) presented above in IV, Section C above and the limits are taken with respect to time.

IV. Conclusion

The theoretical considerations for computing Lyapunov exponents of continuous and discrete dynamical systems are presented and extended to hybrid dynamical systems. A numerical example demonstrating implementation of the methods developed is presented.

There are several benefits to using the hybrid systems approach. While the theory of Lyapunov exponents is well-developed for both continuous and discrete time dynamical systems, little work has been done concerning hybrid systems. Using the approach presented in this paper, a theoretical framework has been developed that can be extended to other hybrid systems. Conventionally, Lyapunov functions have been used for hybrid system stability analysis [7]. The analysis of Lyapunov exponents provides another approach for analyzing the stability of hybrid dynamical systems.

Much follow-on work would be beneficial to the development of these ideas to hybrid dynamical systems. For example, many examples in [7] show multiple solutions ϕ to a hybrid dynamical system with the same initial condition. The theory of Lyapunov exponents presented in this work requires that the solution to ϕ be unique in order for the exponents to be a meaningful global measurement.

V. Acknowledgments

The author gratefully acknowledges the continuing collaboration on this work with his advisor, Joceline Lega, of the Department of Mathematics at the University of Arizona.

References

[1] Oseledec, V.I. A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems. *Trans. Mosc. Math. Soc*, 19:197, 1968.

[2] Forster, C. *Lyapunov Instability of Two-Dimensional Fluids*. Master's thesis, University of Vienna, May 2002.

[3] Ott, E. *Chaos in dynamical systems*. University Press, Cambridge, 1993.

[4] Dellago, Ch., H.A. Posch, and W.G. Hoover. Lyapunov instability in a system of hard disks in equilibrium and nonequilibrium steady states. *Phys. Rev. E*, 53(2):1485, 1996.

[5] Benettin, G., L. Galgani, A. Giorgilli, and J.-M. Strelcyn. *Meccanica* **15**, 9 (1980).

[6] Ruelle, D. *Chaotic evolution and strange attractors*. University Press, Cambridge, 1989.

[7] Goebel, R., R.G. Sanfelice, and A.R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. AME549 Lecture Notes. Available at *https://d2l.arizona.edu/d2l/lms/content/home.d2l?ou=221128*, 2011.